Mapping Classes as Braids

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Abstract

Among the many developments which one can associate to the rather ambitious title of this review paper, I aim only to explain two basic things:

1. Why the mapping class group (MCG) for a disk with n-punctures is the braid-group on n strands
2. How to understand the braiding relations for Dehn twists in a closed orientable surface in terms of braid groups.

1 Braids and Maps of Punctured Disks

1.1 Marked Surfaces

Let Σ be connected compact orientable surface possibly with boundary. Let $Homeo^+(\Sigma)$ denote the set of orientation preserving homeomorphisms on Σ rel $\partial\Sigma$, equipped with the compact-open topology. The mapping class group counts path-connected components in this space:

$$MCG(\Sigma) = \pi_0(Homeo^+(\Sigma)) \tag{1}$$

Under the compact-open topology a continuous path from f to g is an isotopy, so equivalently mapping class group is the group of all homeomorphisms rel boundary up to isotopy.

The central objects of the present discussion is a slight generalization of this, obtained by removing points from the surface and considering homeomorphisms which permute these punctures.

Definition 1. *Take a set of n distinguished points $P \equiv \{x_1 \cdots, x_n\} \subset \Sigma$, let $Homeo^+(\Sigma, P)$ denote the space of homeomorphisms which preserve this set of marked points*

$$Homeo^+(\Sigma, P) \equiv \{f \in Homeo^+(\Sigma) \mid f(P) = P\} \tag{2}$$

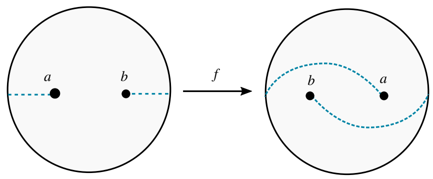
The mapping class group with marked points P is the set of connected components in this space [5]

$$MCG(\Sigma, P) \equiv \pi_0(Homeo^+(\Sigma, P)) \tag{3}$$

In words, this says the mapping class group of a surface with marked points are homeomorphisms (rel boundary) which preserve the set marked points. up to isotopies which also preserves this set of points.

Example: the twice-punctured disk The mapping class group with marked points in general look very different from the mapping class group without marked points. To see this, consider the example of a closed disk $\Sigma \equiv D^2$. The Alexander trick gives one way to construct an isotopy from any homeomorphism of the disk to the identity, so the mapping class is trivial $MCG(D^2) = \{1\}$.

Now mark two points a, b, and consider a boundary-preserving homeomorphism f which swaps a and b.



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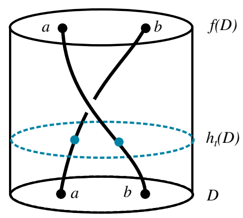
Of course there exists an isotopy h_t from f to the identity, e.g. by Alexander's trick. Observe any such isotopy will have to somehow continuously drag the point a to $f^{-1}(a) = b$, like so



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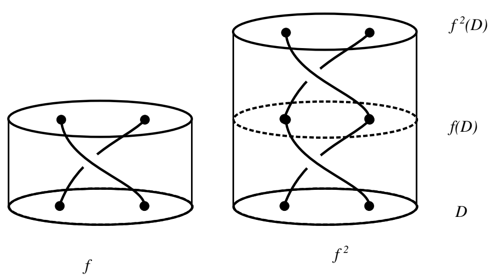
It is clear that at some time t in the isotopy, h_t will have to fail to preserve the marked points $h_t(a) \notin \{a, b\}$. Hence f is not related to the identity via an isotopy preserving the marked points, therefore we have exhibited an non-trivial element of $MCG(D^2, \{a, b\})$.

There are a few other interesting properties of this mapping class group:

1. A composition of the twist f with itself gives another inequivalent mapping class in $MCG(D^2, \{a, b\})$. The corresponding isotopy to Id looks like:



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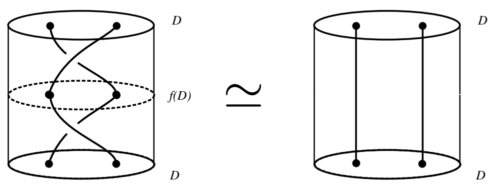
2. The inverse f^{-1} is another twist.



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Observe these figures look like pictures of 2 strings braided together. Furthermore since we only care about maps up to isotopy, any such "braiding diagram" like the ones above determines a unique element of $MCG(D^2, \{a, b\})$. It is now natural to ask whether all mapping classes for the twice-punctured torus generated this way - the answer turns out to be yes. Based on this intuition, it is also natural to conjecture that the mapping class group of a disk with n marked points correspond to ways to "braid n strands" - this also turns out to be true. The remainer of section 1 formalizes this intuition and proves these statements.

We also note that figures drawn above should also be reminiscent of what happens in a neighborhood of a simple closed curve when one performs a Dehn twist around it. This relationship is made precise in section 2 of this article.

1.2 The Birman Sequence

We just saw that MCG for surfaces with marked points in general look very different from MCGs for surfaces without marked points. It is natural to ask how they're related. In fact one can broaden this question slightly and consider the marked and unmarked homeomorphism groups. One can also simplify this question slightly by starting with the case of a single marked point p.

To this end, one way to think about the marked and unmarked homeomorphism groups is the following fibration. First consider the projection

$$Homeo^+(\Sigma) \xrightarrow{\pi} \Sigma \tag{8}$$

where the projection map is the evaluation of the homeomorphism at the marked point p

$$\pi : f \mapsto f(p) \tag{9}$$

The fiber over a point looks like

$$\pi^{-1}(q) = \{f \mid f(p) = q\} \simeq \{f \mid f(p) = p\} = Homeo^+(\Sigma, p) \tag{10}$$

where $\simeq$ denotes isotopy - any f that fixes $f(p) = q$ can be isotoped such that q is dragged to p. It is straightforward to exhibit the local product structure of the fibration, and that we have a fiber bundle

$$Homeo^+(\Sigma, p) \hookrightarrow Homeo^+(\Sigma) \xrightarrow{\pi} M \tag{11}$$

This bundle induces a long exact sequence of homotopy groups (see, e.g. chapter 4 of Hatcher [6]).

$$\cdots \to \pi_n(Homeo^+(\Sigma, p)) \to \pi_n(Homeo^+(\Sigma)) \to \pi_n(\Sigma) \to \pi_{n-1}(Homeo^+(\Sigma, p)) \to \cdots \tag{12}$$

In particular, the portion at the end of the sequence looks like

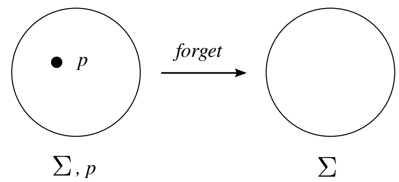
$$\pi_1(\Sigma) \to \pi_0(Homeo^+(\Sigma, p)) \to \pi_0(Homeo^+(\Sigma)) \to \pi_0(\Sigma) \tag{13}$$

Recalling the definition of the mapping class groups, this reads

$$1 \to \pi_1(\Sigma) \to MCG(\Sigma, p) \to MCG(\Sigma) \to 1 \tag{14}$$

This is known in literature as the Birman exact sequence [3], which relates the mapping class groups of the marked and unmarked surfaces. It is helpful to have some more explicit ways to think about the arrows in this sequence

1. The map $MCG(\Sigma, \{x_1, \cdots, x_n\}) \to MCG(\Sigma)$ clearly is just a map that "forgets" the marked point. Picture this as the map on mapping class group induced by filling in a puncture p



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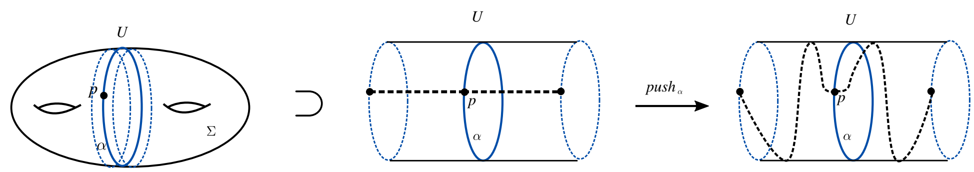
2. The map $\pi_1(\Sigma) \to MCG(\Sigma, p)$ takes a loop α to a homeomorphism, by exactness of the next arrow the image homeomorphism must be isotopic to the identity. One way to construct such a map is to "drag" the marked point p along α



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this is known as the "push" map.

This construction generalizes straightforwardly to the case with multiple marked points. In this case, the base space of our fiber bundle is the configuration space of n distinct points in M, which we write as $C_n\Sigma$ (we will study this space more closely in a later section). The fibration is

$$Homeo^+(\Sigma, \{x_1, \cdots, x_n\}) \hookrightarrow Homeo^+(\Sigma) \xrightarrow{\pi} C_n\Sigma \tag{17}$$

And we get the exact sequence

$$1 \to \pi_1(C_n\Sigma) \to MCG(\Sigma, \{x_1, \cdots, x_n\}) \to MCG(\Sigma) \to 1 \tag{18}$$

This is the Birman exact sequence for n-punctures.

An immediate application of the Birman sequence will justify its presence in the present paper. Consider again the case of the disk D^2. The mapping class group of the unmarked D^2 is trivial, exactness of the Birman sequence then forces

$$\pi_1(C_n\Sigma) \simeq MCG(\Sigma, \{x_1, \cdots, x_n\}) \tag{19}$$

As before, the isomorphism should be thought of as a point-dragging map which drags the marked points along a loop in the configuration space of points. Such a loop looks very much like a braid - we discuss precisely how this works now.

1.3 Braids

Motivated by our discussion of mapping class groups with marked points in section 1.1, we want to formalize the notion of "ways to braid n strands". This is exactly the origin of the notion of the braid group [7].

Definition 2. *Fix some distinct points $x_1, \cdots, x_n \in \mathbb{R}^2$.*

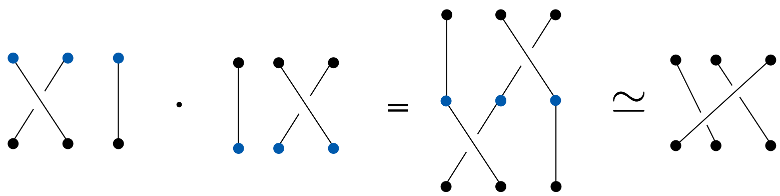
A ***braid on*** n ***strands*** *is the isotopy class of a collection of non-intersecting paths $\gamma_i : [0,1] \to [0,1] \times \mathbb{R}^2, i \in 0 \cdots n$ such that*

$$\gamma_i(0) = x_i, \gamma_i = x_{\rho(i)} \tag{20}$$

for some permutation $\rho \in S_n$ of $1 \cdots n$.

The braid group on n strands *B_n is the set of all braids on n strands, the group operation is composition of paths, the inverse operation is reverse-traversal of paths.*

Example For example, here's a composition of 2 braids on 3 strands



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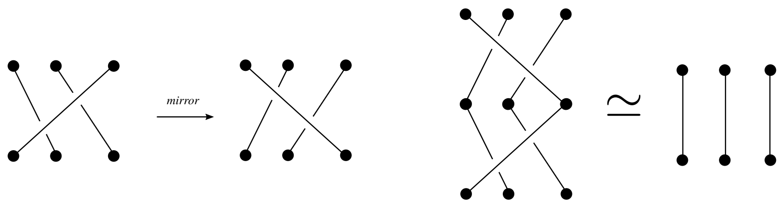
and inverse



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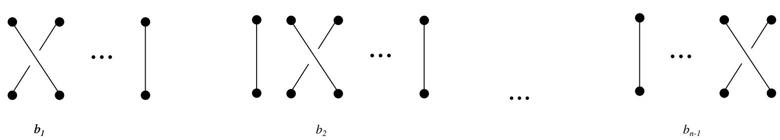
x

Generators It is easy to see that there is a set of $n-1$ generators for B_n



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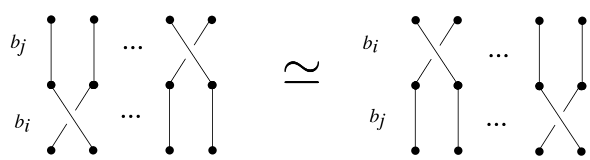
to see that this indeed generate the group, simply take any braid, then then arrange for each "crossing" to occur at a different time t.

It is clear that braid generators on different pairs of strands commute



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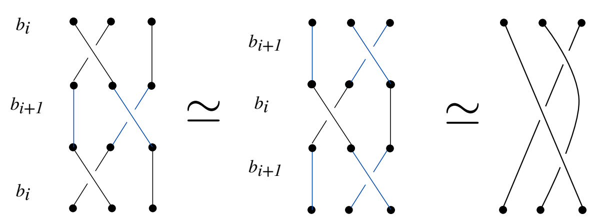
It is natural to ask how braids adjacent threads interact, observe



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These are known as the braid relations, which together give a presentation of the group

$$B_n = \langle b_1 \cdots b_{n-1} | \; b_{i+1} b_1 b_{i+1} = b_i b_{i+1} b_i, b_i b_j = b_j b_i \quad |i-j| \geq 2 \rangle \tag{26}$$

Braids as Configuration Spaces As promised earlier, we now show that the fundamental group of the configuration space of points that show up in the Birman sequence is a braid group. First let's give a more precisely of $C_n\Sigma$ as a manifold. One takes the product of Σ with itself n times, remove the configurations where any two points coincide, and quotient by the permutations of points because any two configurations related by a permutation is considered equivalent.

$$C_n\Sigma \equiv (\Sigma^{\times n} - \{(p_1, \cdots, p_n) \in \Sigma^{\times n} \mid p_i = p_j\})/S_n \tag{27}$$

where S_n is the symmetric group (permutations of $1, \cdots n$) acting on the product manifold by the obvious permutation. Consider a representative loop $\alpha \in \pi_1(C_n(\Sigma))$. $\alpha(t)$ is a continuous path of n point configurations that begin and end at the same points. Hence α traces out a braid on n strands. Furthermore it is clear that two loops are homotopic $\iff$ the braids they trace out can be deformed into eachother. Hence $\pi_1(C_n\Sigma) \simeq B_n$

The Birman sequence tells us how to relate these fundamental groups to mapping class groups, the result of our discussion of Birman sequence applied to a disk now yields

Theorem 1.

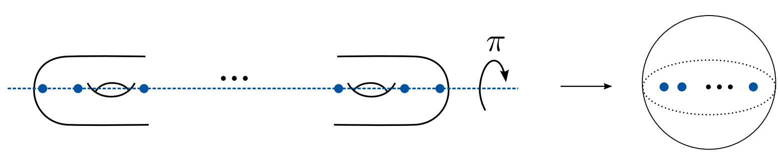
$$B_n \simeq \pi_1(C_n\Sigma) \simeq MCG(\Sigma, \{x_1, \cdots, x_n\}) \tag{28}$$

This is the primary result of this section: mapping class group of n-punctured disk is the braid group on n-strands. Our earlier example for a twice-punctured disk is a special case of this result.

2 Braids and Dehn Twists

So far we saw that braid groups are mapping class groups of punctured disks. It is natural to ask whether braid groups tell us anything about mapping class groups of more interesting surfaces. Indeed the remainder of this article is dedicated to explaining how to use braids to understand Dehn-twists on genus-g surfaces.

There's a natural way to obtain a $2g+2$-punctured sphere from a genus-g surface. This is the quotient by the π- rotation R about a skwer [1, 4], best explained through a picture



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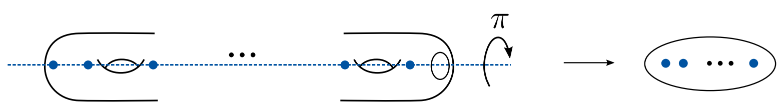
The fixed points of this map are the intersection points of the skewer with the surface. Away from the intersection points the projection to the quotient by R is a double covering map. With the fixed points removed from the domain, we have a double covering map to the $2g+2$-punctured sphere.

In order to apply our braids machinery, we want an closed disk. Simply remove an open disk in the procedure described above



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where the disk downstairs has $2g+1$ punctures (because we took the skewer to go through the removed disk). By our discussion in section 1, the mapping classes of the disk downstairs is given by the braid groups. How much does this tell us about homeomorphisms in the surface upstairs? Observe

1. Any homeomorphism upstairs which commutes with R descends to a homeomorphism downstairs
2. A homeomorphism downstairs has a unique lift upstairs preserving the marked points. This lift necessarily commutes with R.

Hence we conclude the homeomorphisms downstairs tells us about the homeomorphisms upstairs that commute with R. This leads us to the notion of the symmetric mapping class group

Definition 3.

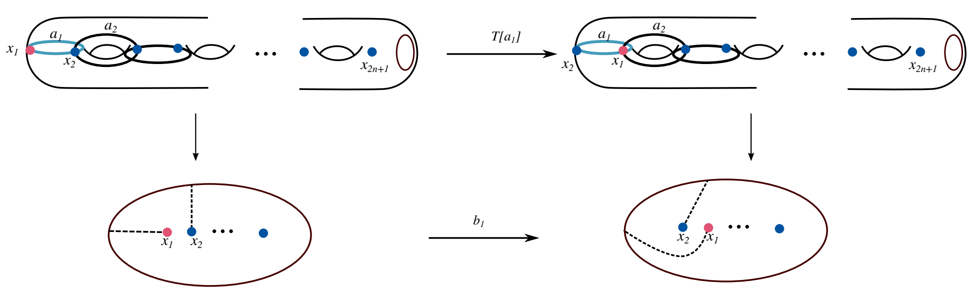
$$SMCG \equiv (Centralizer\ of\ R\ in\ Homeo^{+}(\Sigma))/Isotopy \tag{31}$$

that this group is given by the mapping class group of the disk, as suggested by our quotient, is a theorem due to Birman and Hilden [2]

Theorem 2.

$$SMCG(\Sigma_{2g+1}) \simeq B_{2g+1} \tag{32}$$

The primary examples of elements of these symmetric mapping classes will be Dehn twists. Consider an essential (non-separating) curve γ preserved by R. Then there is some neighborhood U of γ preserved by R. Any Dehn twist about γ can be isotoped such that it is identity on $\Sigma - U$. Therefore Dehn twists around simple closed curves are elements of the symmetric mapping class group. It follows that they should have representations as braids downstairs. To see how this works, simply observe that an essential curve preserved by R must contain two of the points fixed by the skewer. A dehn twist by π about the this curve then swaps these points:



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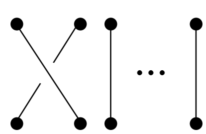
As a braid, the Dehn twist pictured above is therefore the generator b_1



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The braid relations in the braid group now translate directly to statements about compositions of dehn twists. Let $T[\alpha]$ denote Dehn twist about simple closed curve α. Commutativity of non-adjacent strand-pairs is the statement

$$T[\alpha] \circ T[\beta] = T[\beta] \circ T[\alpha] \tag{35}$$

for non-intersecting curves α and β.

The braid relation for adjacent strands is the statement

$$T[\alpha] \circ T[\gamma] \circ T[\alpha] = T[\gamma] \circ T[\alpha] \circ T[\gamma] \tag{36}$$

for linked simple closed curved γ and α. These relations are known as the braid relations for Dehn twists.

References

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