Introduction to Calibrated Manifolds

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Abstract

I give a pedagogical review of calibrated geometry, as pioneered by Harvey and Lawson. After introducing the general notion of calibrations, I will focus on building intuition through a variety examples. In particular I will discuss

1. Simple examples of calibrations in $\mathbb{R}^n$ and $\mathbb{H}^n$
2. How to generate calibrations using Riemannian holonomy

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1.1 Differential Forms on Subspaces

We begin by reviewing some simple definitions to fix notation. For our purposes all manifolds discussed are connected.

Let (M, g) be a Riemannian manifold of dimension n. The Riemannian structure gives rise to a natural volume form on M, furthermore it gives natural volume forms on subspaces of tangent spaces. That is, on each dimension k subspace $V \subset T_pM$, there's a natural Riemannian volume form given by

$$Vol_V \equiv e_1 \wedge \cdots \wedge e_k \tag{1}$$

where $(e_i)_{i \in 1 \cdots k}$ is an orthonormal basis for V^* [1].

Recall that k-form $\omega \in \Omega^k_p(M)$ is a completely skew-symmetric linear map $\omega : T_pM^{\otimes} \to \mathbb{R}$. By restricting a k-form to a subspace V we mean the restriction

[1] Equivalently this can be defined as usual via determinant of the restricted metric $g_p|_V$, equivalence of these definitions follows from a change of basis

$$\omega|_V : V^{\otimes k} \to \mathbb{R} \quad \in \Omega^k(V) \tag{2}$$

By skew-symmetry, for dimension k subspaces $V \subset T_pM$, $\Omega^k(V)$ is a dimension 1 vector space over $\mathbb{R}$. For convenience, let $a, b \in \Omega^k(V)$, we write

$$a \leq b \tag{3}$$

as a comparison of their $\mathbb{R}$ component in any basis.

1.2 Calibrated Submanifolds

Definition 1. *A closed k-form $\varphi \in \Omega^k(M)$ is said to be a **calibration** [4] if*

$$\varphi_p|_V \leq Vol_V \tag{4}$$

for any dimension k subspace $V \subset T_pM$, $\forall p \in M$.

One way to generate these subspaces of T_pM is to consider tangent space of a k-submanifold $T_p\Sigma$. It turns out to be interesting to consider submanifolds where the inequality in **Definition 1** is saturated

Definition 2. *A dimension k submanifold $\Sigma \subset M$ is said to be **calibrated by** φ if*

$$\varphi|_{T_p\Sigma} = Vol_{T_p\Sigma} \quad \forall p \in \Sigma \tag{5}$$

The fundamental property of calibrations is that

Theorem 1. *Calibrated submanifolds are volume-minimizing within their homology class.*

The proof is a one-liner

Proof. Let Σ be submanifold calibrated by φ, Σ' be homologous to Σ.

$$\int_\Sigma Vol_{T\Sigma} = \int_\Sigma \varphi|_{T\Sigma} = \int_{\Sigma'} \varphi|_{T\Sigma'} \leq \int_{\Sigma'} Vol_{T\Sigma'} \tag{6}$$

where

1. The first equality is the statement Σ is calibrated
2. Second equality follows from Stoke's theorem because φ is closed
3. The final inequality is the fact that φ is a calibration.

To be completely clear let us explain the line invoking Stoke's theorem. In the notation of the homology-cohomology pairing

$$\langle \cdot, \cdot \rangle : (\Sigma, \omega) \mapsto \int_\Sigma \omega \tag{7}$$

Observe

$$\langle \Sigma - \Sigma', \omega \rangle = \langle \partial\Omega, \omega \rangle = \langle \Omega, d\omega \rangle = 0 \implies \langle \Sigma, \omega \rangle = \langle \Sigma', \omega \rangle \tag{8}$$

where second to last equality is Sstoke's theorem. The above proof simplifies to

$$\langle [\Sigma], Vol_\Sigma \rangle = \langle [\Sigma], \varphi \rangle \leq \langle [\Sigma], Vol_{\Sigma'} \rangle \tag{9}$$

□

In passing we mention that this volume minimizing property is reminescent of the max-flow-min-cut theorem in graph and network theory. i.e. we obtain a minimal submanifold by maximizing the weight of the restricted calibration form. It is an interesting open question whether there is a precise way to view calibrated geometry as a "smooth version" of max-flow-min-cut. For a discussion of work in this direction see the very recent paper by Michael Freedman [2].

2 Simple Examples

We demonstrate the machinery introduced above with some very simple examples of calibrations from familiar real manifolds. [2]

First, a trivial, universal example

Example 1. $\varphi = 0$ *is a calibration on any manifold* M*, for any* k*. Clearly this trivial calibration has no calibrated submanifolds.*

Now we move on to progressively more interesting examples.

2.1 Calibrations on the Euclidean Plane

Let's find volume-minimizing dimension 1 submanifolds of $M \equiv \mathbb{R}^2$ with the standard Euclidean metric $\langle , \rangle$.From elementary school we know that these are straight lines. Turns out we can discover this fact through calibrated geometry.

Our proof for the volume-minimization property was for compact manifolds. Take, say, the stereographic projection which maps onto plane to the sphere with one point deleted. There exists a metric on the sphere such that the stereographic projection is an isometry onto its own image. The sphere has trivial first-homology class, so it makes sense to talk about volume minimizing curves on the sphere with respect to this Euclidean metric. By our discussion above one way to find some of these volume minimizing curves is through calibrations.

Take (x, y) to be the standard coordinates on $\mathbb{R}^2$. Let $p \in M$, a dimension 1 submanifold is given by the span of a vector:

$$V_{\alpha,\beta} = span(\alpha\partial_x + \beta\partial_y) \subset T_pM \tag{10}$$

we're free to normalize this basis vector such that $\alpha^2 + \beta^2 = 1$, hence our subspaces are parametrized by a single parameter α:

$$V_\alpha \equiv span(\alpha\partial_x + \sqrt{1-\alpha^2}\partial_y) \tag{11}$$

$$\{\text{Linear subspaces of } T_pM\} = \{V_\alpha \mid \alpha \in (-1, 1]\} \tag{12}$$

Observe $V_{-1} = V_1$, so topologically our set of subspaces is a circle. Another way to see this is recall the set of linear subspaces of $\mathbb{R}^2 \simeq T_pM$ is the real projective line $\mathbb{PR}^1$, obtained by quotienting the unit circle by the antipodal map, which in turn yields another circle - $\mathbb{PR}^1 \simeq S^1$.

The dual subspace V_α^* is given by

$$V_\alpha^* = span\{\alpha dx + \sqrt{1-\alpha^2}dy\} \subset T_p^*M \tag{13}$$

[2]Such examples appear to be lacking in literature, perhaps because they are too trivial.

The basis 1-form $\hat{e}(\alpha) \equiv \alpha dx + \sqrt{1-\alpha^2}dy$ is "unit" in the sense that it is metric dual to a unit vector. That is

$$\langle e(\alpha), \cdot\rangle \equiv \hat{e}(\alpha)[\cdot] = \langle \alpha\partial_x + \sqrt{1-\alpha^2}\partial_y, \cdot\rangle \tag{14}$$

$$\langle e(\alpha), e(\alpha)\rangle = \alpha^2 + (\sqrt{1-\alpha^2})^2 = 1 \tag{15}$$

The volume for is the wedge product of an orthonormal basis, in this case a basis of 1 element:

$$Vol_{V_\alpha} = \hat{e}(\alpha) = \alpha dx + \sqrt{1-\alpha^2}dy \tag{16}$$

Now consider a general 1-form $\varphi \in T^*M$:

$$\varphi_p = \sigma(p)dx + \tau(p)dy, \quad \sigma, \tau \in C^\infty(M) \tag{17}$$

Compute the restriction of this form to a subspace. It is sufficient to evaluate it on the basis vector

$$\varphi[e(\alpha)] = \varphi[\alpha\partial_x + \sqrt{1-\alpha^2}\partial_y] = \sigma\alpha + \tau\sqrt{1-\alpha^2} = (\sigma\alpha + \tau\sqrt{1-\alpha^2})\hat{e}(\alpha)[e(\alpha)] \tag{18}$$

Hence we found

$$\varphi = (\sigma\alpha + \tau\sqrt{1-\alpha^2})\hat{e} = (\sigma\alpha + \tau\sqrt{1-\alpha^2})Vol_\alpha \tag{19}$$

We can now ask when is φ a calibration form. It turns out we don't even need to answer this question in general, but rather just need to find some simple cases when φ happens to be a calibration.

This simple case is when σ, τ are constant - then all derivatives vanish, so $d\varphi = 0$. Now by our equation relating φ_α to Vol_α, φ is a calibration if and only if

$$\sigma\alpha + \tau\sqrt{1-\alpha^2} \leq 1 \tag{20}$$

1-forms which satisfy this yield an abundance of examples of calibrations.

Example 2. $\sigma = 1, \tau = 0 \implies \varphi = dx$. *Manifolds calibrated by this have tangent spaces spanned by basis vectors* $e(\alpha)$ *which saturate the above inequality, which in this case simply reads*

$$\alpha = 1 \implies e(\alpha) = \partial_x \tag{21}$$

this is the distribution defined by φ. *Integral manifolds are integral curves, which are horizontal lines.*

By an identical computation, $\varphi = dy$ *calibrate vertical lines.*

This is a special case of the following more general result

Example 3. *Fix some* $\sigma \in (-1, 1], \tau = \sqrt{1-\sigma^2}$. *One can check by caluculus that the function*

$$\sigma\alpha + \sqrt{1-\sigma^2}\sqrt{1-\alpha^2} \tag{22}$$

as a single-variable function in α *has global maximum value* $\sqrt{\sigma^2+\tau^2} = 1$. *Hence the inequality above is satisfied, hence* $\varphi(\sigma) \equiv \sigma dx + \sqrt{1-\sigma^2}dy$ *is a calibration. The inequality is saturated when*

$$\sigma\alpha + \sqrt{1-\sigma^2}\sqrt{1-\alpha^2} = 1 \tag{23}$$

Observe this happens when $\alpha = \sigma$. *The subspaces are spanned by vectors* $e = \sigma\partial_x + \sqrt{1-\sigma^2}\partial_y$, *whose integral curves are lines along the unit vector* $(\sigma, \sqrt{1-\sigma^2})$.

Hence, we found a parametrized family of calibrations $\varphi(\sigma), \sigma \in S^1$, *each calibration determines a calibrated foliation of parallel lines. It turns out these are all the volume-minimizing 1-manifolds in* $\mathbb{R}^2$.

Hence we found calibrations whose corresponding calibrated submanifolds are straight lines on the plane. As we know these turn out to be all volume-minimizing 1-manifolds on the plane that are mapped to cycles on the sphere. One might be optimistic about always being able to find all volume-minimizing submanifolds through calibrations, and that calibrations can always be constructed with such ease. The following example will show that this is not the case.

2.2 Calibrations on the Hyperbolic Plane

Consider the hyperbolic plane $M \equiv \mathbb{H}^2 \equiv (\mathbb{R} \times \mathbb{R}^+, g_{\mathbb{R}^2}/y^2)$. A unit vector in $T_{(x,y)}M$ looks like

$$e \equiv \alpha \partial_x + \sqrt{y^2 - \alpha^2}\partial_y \tag{24}$$

for some $\alpha \in [-y, y)$. It is a unit vector in the sense that

$$\langle e, e \rangle = \frac{1}{y^2}(\alpha^2 + y^2 - \alpha^2) = 1 \tag{25}$$

The dual vector is given by

$$\hat{e} \equiv \langle e, \cdot \rangle = \frac{1}{y^2}(\alpha dx + \sqrt{y^2 - \alpha^2} dy) \tag{26}$$

Again now take a general one-form

$$\varphi = \sigma dx + \tau dy, \quad \sigma, \tau \in C^\infty(M) \tag{27}$$

Compute

$$\varphi[e] = (\sigma\alpha + \tau\sqrt{y^2 - \alpha^2}) = (\sigma\alpha + \tau\sqrt{y^2 - \alpha^2})\hat{e}[e] \tag{28}$$

$$\implies \varphi = (\sigma\alpha + \tau\sqrt{y^2 - \alpha^2})\hat{e} \tag{29}$$

As before, on our 1-manifold, the unit vector $\hat{e}$ is the Riemannian volume form, so in order for φ to be a calibration we need

1. $$d\varphi = 0 \tag{30}$$

2. $$\sigma\alpha + \tau\sqrt{y^2 - \alpha^2} \leq 1 \tag{31}$$

It should be now apparent that forms like these are harder to find than the analog problem in the Euclidean case. We're only guarenteed $|\alpha| \leq y$, in order for the inequality to hold we need something to the effect of $\sigma, \tau \sim \frac{1}{y}$ so that LHS has hope of being suppressed by RHS for arbitrary y.

In particular observe no constant calibration will satisfy this, which means we need to consider general functions which satisfy the inequality, then demand the closedness of φ, which gives some simple partial differential equation. Because this is rather tedious, we only give one example of a calibration on the hyperbolic plane.

Example 4. $\varphi = \frac{1}{y}dy$. *Observe* $d\varphi \propto dy \wedge dy = 0$*, and the inequality becomes*

$$\frac{\sqrt{y^2 - \alpha^2}}{y} = \sqrt{1 - \left(\frac{\alpha}{y}\right)^2} \leq 1 \tag{32}$$

which certainly holds because $|\alpha| < y$*, so* φ *is a calibration. Calibrated submanifolds have tangent spaces spanned by unit vectors with* $\alpha = 0 \implies e = \partial_y$*. Integral curves are vertical lines.*

In this example we saw that in general, calibrations are not so easy to find. [3]This motivates us to think about general ways to generate calibrations. In the next section we show how to use the holonomy group to do this.

3 Calibrations from Holonomy

Motivation One take-away from our examples above is that the "lighter than volume form" property is easy to satisfy on each tangent space. That is, at any particular tangent space T_pM, one can always scale a non-zero form $\varphi \in \Omega_p^k M$ such that $\varphi|_V \leq Vol|_V$. This scaling factor is non-zero because $Vol|_V$ does not vanish on any subspace V, because the metric is positive-definite.

Hence, finding calibrations become difficult only when we insist on finding smooth closed forms which satisfy the above local property. It is then interesting to ask which forms defined at a point have extensions to calibration forms over all of M. It turns out that the local forms invariant under action of the holonomy group on M admit "constant extensions", which are closed form fields. We discuss how this works now.

3.1 The Holonomy Group

We briefly review the notion of holonomy.

Connections & Parallel Transport The Riemannian structure on (M, g), by Levi-Civita theorem, gives a unique Levi-Civita connection ∇. There's a path-lifting property associated with ∇:

Proposition 1. *Given a piecewise C^1 curve $\gamma : [0,1] \to M$, and vector $v \in T_{\gamma(0)M}$, there is a unique parallel vector field $\tilde{\gamma}^{(v)}$ along γ such that $\tilde{\gamma}_*^{(v)} \frac{d}{d}|_0 = v$.*

For proof, consult any Riemannian geometry book, e.g. [4]. This then gives a notion of parallel transport:

$$P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M, \quad v \mapsto \gamma^{(v)}(1) \tag{33}$$

It can be checked that as a map between vector spaces, the parallel transport map is linear and degenerate. Furthermore these maps compose under concatenation of curves. Let γ, σ be curves such that $\sigma(0) = \gamma(1)$, let $\sigma \circ \gamma$ denote the concatenated curve. It follows easily from the above definition that

1. $P_\sigma \circ P_\gamma = P_{\sigma \circ \gamma}$
2. $P_\gamma \circ P_{\gamma^R} = Id_{T_{\gamma(0)}M}$

where $\gamma^R(t) \equiv \gamma(1-t)$ is the reversely-parametrized curve.

Holonomy A interesting special case is the case where γ is a loop, i.e. $\gamma(0) = \gamma(1) = p$. In this case P_γ becomes simply a linear automorphism on T_pM. Pictorially the non-triviality of P_γ reflects the failure of the lifted curves $\tilde{\gamma}$ to be closed loops in TM.

The holonomy group at T_pM is the collection of all linear maps on T_pM one can generate by parallel transporting along loops Let L_pM denote the pointed loop space (viewed only as a group for our purposes)

[3]It does turn out however that every codimension 1 manifold is calibrated. This non-trivial result was proved by Freedman in an appendix of his very recent paper [2]

$$L_pM \equiv \{\gamma| \ \gamma(0) = \gamma(1) = p\} \tag{34}$$

L_pM is a group under concatneation of curves (concatneation of loops are loops, loop with parametrization is inverse). The holonomy group is given in terms of the loops space as follows

$$Hol_pM \equiv \{P_\gamma| \ \ \gamma \in L_pM\} \tag{35}$$

By the composition properties for parallel transport discussed ealier, L_pM is a group implies Hol_pM is a group.

Base point independence An isomorphism of pointed loop spaces is given by conjugation by a connecting curve. Let c be curve such that $c(0) = p, c(1) = q$, we have a group isomorphism

$$L_pM \simeq c^R L_q c \tag{36}$$

Which translates to a isomorphism of the holonomy groups

$$Hol_p(M) \simeq P_{c^R} Hol_q(M) P_c \tag{37}$$

Hence the holonomy group at any p are isomorphic to the holonomy group at any other point q, therefore it makes sense to say "the holonomy group of M":

$$Hol(M) \equiv Hol_pM \simeq Hol_qM \ \ \forall p, q, \in M \tag{38}$$

3.2 Calibrations from Holonomy

Parallel Tensors Let $\omega \in T^{(m,n)}M \equiv (\otimes^m TM) \otimes (\otimes^n TM)$ be a rank (m, n) tensor field. ω is said to parallel if

$$\nabla \omega = 0 \tag{39}$$

The generalization of the lifting property for vector fields to tensor fields is

Proposition 2. *Let $\gamma : [0, 1] \to M$ be piecewise C^1 curve. Let $v \in T_p^{(m,n)}M$. There exists a unique parallel tensor field along γ, $\tilde{\gamma}^{(v)}$, such that $\tilde{\gamma}^{(v)}(0) = v$.*

The notion of parallel transport of tensors is therefore well defined. A simple observation is

Proposition 3. *Parallel tensors are invariant under action of the holonomy group, that is*

$$\nabla \omega = 0 \implies Hol_p(M)(\omega|_p) = \omega_p \tag{40}$$

Proof. Let $P_\gamma \in Hol_pM$. $\omega \circ \gamma$ is a lift of γ to $T^{(m,n)}M$. Furthermore $\omega \circ \gamma(0) = \omega|_p$ and $\nabla \omega = 0$, so $\omega \circ \gamma$ is the unique parallel lift of γ with initial value $\omega|_p$. Then by the defintiion of parallel transport

$$P_\gamma(\omega|_p) = \omega_{\gamma(1)} = \omega_p \tag{41}$$

□

More important to us is the converse

Proposition 4. *Tensor ω_0 over T_pM invariant under $Hol_p(M)$ admit a unique extension to a parallel tensor field defined on all M, $\omega \in \Omega^k(M)$.*

Proof. Define the extension ω by

$$\omega|_q = P_{\gamma_q}\omega_0 \tag{42}$$

where γ_q is any path starting at p, ending at q. This is independent of the choice of the path γ_q. To see this let a, b be paths between p and q. Then $b^R \circ a$ is a loop based at p, hence $P_{b^R \circ a} \in Hol_p(M)$. But ω_0 is invariant under the holonomy group by assumption, so

$$\omega_0 = P_{b^R \circ a}\omega_0 = P_{b^R} \circ P_a \ \omega_0 \tag{43}$$

$$\implies (P_{b^R})^{-1}\omega_0 = P_a\omega_0 \tag{44}$$

$$\implies P_b\omega_0 = P_a\omega_0 \tag{45}$$

□

Building Calibrations Let $\omega \in \Omega_p^k M$ be a k-form with property

$$\omega|_V \leq Vol|_V \tag{46}$$

for any dimension k subspace $V \subset T_pM$. As discused above any arbitrary form in $\Omega_p^k M$ can be scaled to satisfy this bound.

Now take ω is invariant under $Hol_p(M)$. By a proposition above this implies there is a unique parallel extension φ of ω.

Proposition 5. *φ is a calibration*

Proof. Since φ is parallel, $\nabla\varphi = 0 \implies d\varphi = 0$, so φ is closed.

Since the metric tensor is parallel [1], parallel transport preserves norm, hence

$$\varphi_q|_W = P_\gamma\omega \leq Vol_W \ \forall \text{ dim } K \text{ subspaces } W \subset T_qM \tag{47}$$

where γ is some curve with $\gamma(0) = p, \gamma(1) = q$.

□

It turns out that there is a finite number of possibilities of what the holonomy group for a Riemannian manifold can be. This is the famous classification due to Berger. Applying the above procedure to find calibrations corresponding to each of the possibilities yields a variety of interesting submanifolds in complex geometry. We refer to ref. [3] for a discussion of that.

References

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